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Operation Associated to Partitions and Best Extension of Signed Measures on Set Logics

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We offer a new operation on the class of concrete logics based on a partition generated by a two-valued state. Then we discuss the problem of the best extension of signed measures on the concrete logics and give a series of open problems.

1. AN OPERATION ASSOCIATED TO PARTITIONS OF QUANTUM LOGICS

Let E be a quantum logic (orthomodular poset, OMP) (Gudder, 1979; Kalmbach, 1983), and $S_2(E)$ the set of all two-valued finitely additive states on E. Consider a maximal, with respect to inclusion, subset $D \subset E\backslash\{0, 1\}$ satisfying:

(i) For all $a, b \in D$, $(a \not\perp b)$.

(ii) If $a, b \in E\backslash\{0, 1\}$, $a \perp b$ and $a \vee b \in D$, then either $a \in D$ or $b \in D$.

Such a subset exists by the Zorn lemma. Put $E_1 = D \cup \{1\}$, $E_0 = \{a' \mid a \in E_1\}$. Obviously $E_0 \cap E_1 = \varnothing$.

Theorem 1.1. The following conditions are equivalent:

(1) $E = E_0 \cup E_1$.

(2) For all $a \in E$, $(a \notin E_0 \Leftrightarrow a' \in E_0)$.

(3) There exists $f \in S_2(E)$ such that $E_1 = \{a \in E \mid f(a) = 1\}$.

Proof. (1) $\Rightarrow$ (2): If $a \notin E_0$, then $a \in E_1$ and it follows that $a' \in E_0$. Conversely, if $a' \in E_0$, then $a \in E_1$ and $a \notin E_0$ because $E_0 \cap E_1 = \varnothing$.

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(2) $\Rightarrow$ (1): Let $x \in E \backslash E_0 \cup E_1$. We have $x \notin E_0$, $x' \in E_0$. Definition of E_0 gives us $x \in E_1$. This is a contradiction.

(1) $\Rightarrow$ (3): Put

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in E_1 \\ 0 & \text{if} \quad x \in E_0 \end{cases}$$

Let us verify that $f \in S_2(E)$. Assume $a, b \in E\backslash\{0, 1\}$ and $a \perp b$. Consider the case $a \vee b \in D$. By condition (ii) we have either $a \in D$ or $b \in D$. If $a \in D$, then $b \notin D$. It follows that $b \notin E_1$, $b \in E_0$, and $f(a \vee b) = 1, f(a) = 1, f(b) = 0$. If $a \vee b = 1$, then $b = a'$ and the statement is obvious.

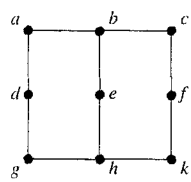
Consider now the case $a \vee b \in E_0\backslash\{0\}$. We have $(a \vee b)' \in D$ and $a \perp (a \vee b)'$, $b \perp (a \vee b)'$. It follows by (i) that $a \notin D$, $b \notin D$, or $a \notin E_1$, $b \notin E_1$. Consequently we see that $a, b \in E_0$ and $f(a \vee b) = f(a) = f(b) = 0$.

(3) $\Rightarrow$ (1): Let $f \in S_2(E)$. Denote by $D = \{a \in E | f(a) = 1\}\backslash\{1\}$. We show that D is a maximal subset of $E\backslash\{0, 1\}$ satisfying (i), (ii). Assume $x \in E\backslash\{0, 1\}$ and $x \notin D$. Then we have $f(x) = 0, f(x') = 1$, and consequently $x' \in D$. Thus we cannot extend D by adding an element x without a breach of condition (i). The rest is straightforward. This finishes the proof.

Remark 1.2. It is easily seen that statement (3) of the Theorem 1.1 implies the heredity of D:

$$a \in D, \qquad b \in E\backslash\{0, 1\}, \qquad a \leq b \Rightarrow b \in D \tag{*}$$

There exists an OMP E with $S_2(E) \neq \emptyset$ and with a maximal subset D satisfying (*), but which does not form the partition of E: consider the OMP E with the following Greechie diagram (Greechie, 1971)



and put $D = \{a', b', c', d, e, f, g', h', k'\}$. We also offer the analog of (*) for E_0:

$$a \in E_0, \qquad b \in E, \qquad b \leq a \Rightarrow b \in E_0 \tag{**}$$

Now, suppose that E is the concrete logic (c.l.) with the support X and F is the concrete logic with support Y. Let $p = \{F_0, F_1\}$ be a partition of F as in Theorem 1.1. For every $Z \subset X \times Y$ consider the section $S_x(Z) = \{y \in Y | (x, y) \in Z\}$ in the point $x \in X$. Put $P_i(Z) = \{x \in X | S_x(Z) \in F_i\}$, $i = 0, 1$.

Theorem 1.3. The collection $E \bowtie F_p = \{Z \subset X \times Y \mid P_0(Z) = P_1(Z)^c \in E\}$ is a concrete logic on the Cartesian product $X \times Y$ which contains all sets of the form $A \times B$, $A \in E$, $B \in F$.

Proof. First we make two observations:

(a) Since $S_x(Z^c) = S_x(Z)^c$, we have $P_0(Z^c) = \{x \in X \mid S_x(Z^c) \in F_0\} = \{x \in X \mid S_x(Z) \in F_1\} = P_1(Z)$ and $P_1(Z^c) = P_0(Z)$ for all $Z \subset X \times Y$.

(b) Consider the condition $P_0(Z) = P_1(Z)^c$ more carefully. If x is a point such that $S_x(Z) \notin F$, then $S_x(Z) \notin F_1$ and hence $x \in P_1(Z)^c$. Therefore $x \in P_0(Z)$ and $S_x(Z) \in F_0 \subset F$. Thus, if $Z \in E \times F_p$, then all sections $S_x(Z) \in F$ for any point $x \in X$!

Let now $Z \in E \bowtie F_p$. Then $P_0(Z^c) = P_1(Z) = P_0(Z)^c \in E$, $P_1(Z^c) = P_0(Z) = P_1(Z)^c \in E$. It follows that $P_0(Z^c) = P_1(Z) = [P_1(Z)^c]^c = [P_1(Z^c)]^c \in E$ and $Z^c \in E \bowtie F_p$.

Suppose $N, M \in E \bowtie F_p$ and $N \cap M = \emptyset$. Then clearly $S_x(N) \cap S_x(M) = \emptyset$ for all $x \in X$. If $x \in P_1(N) \cap P_1(M)$, then $S_x(N) \in F_1$, $S_x(M) \in F_1$ and there is a contradiction with condition (i). This shows that $P_1(N) \cap P_1(M) = \emptyset$ and $P_1(N) \cup P_1(M) \in E$.

By observation (b), $S_x(N), S_x(M) \in F$; hence $S_x(N) \cup S_x(M) \in F$. If $x \in P_1(N)$, that is, $S_x(N) \in F_1$, then from (*) we have $S_x(N) \cup S_x(M) = S_x(N \cup M) \in F_1$, $x \in P_1(N \cup M)$. We obtain $P_1(N) \cup P_1(M) \subset P_1(N \cup M)$. Conversely $P_1(N \cup M) = \{x \in X \mid S_x(N \cup M) = S_x(N) \cup S_x(M) \in F_1\} \subset P_1(N) \cup P_1(M)$ because $S_x(N) \perp S_x(M)$ and condition (ii) is satisfied. Thus we have showed that $P_1(N) \cup P_1(M) = P_1(N \cup M)$.

Now $P_1(N \cup M)^c = P_1(N)^c \cap P_1(M)^c = P_0(N) \cap P_0(M)$ and from (**) we have $P_0(N \cup M) = \{x \in X \mid S_x(N) \cup S_x(M) \in F_0\} \subset P_0(N) \cap P_0(M)$. Hence $P_0(N \cup M)^c \supset P_1(N \cup M)$. Conversely, if $x \in P_0(N \cup M)^c$, that is, $x \notin P_0(N \cup M)$, then $S_x(N) \cup S_x(M) \notin F_0$. It follows that $S_x(N) \cup S_x(M) \in F_1$ and hence $x \in P_1(N \cup M)$. Thus $P_0(N \cup M)^c = P_1(N \cup M) \in E$ and $N \cup M \in E \times F_p$. We have showed $P_0(N \cup M) = P_0(N) \cap P_0(M)$ also.

Finally, we observe that if $A \in E$, $B \in F$, then

$$S_x(A \times B) = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \in A^c \end{cases}$$

$$P_0(A \times B) = \begin{cases} A^c & \text{if } B \in F_1 \\ X & \text{if } B \in F_0 \end{cases}$$

$$P_1(A \times B) = \begin{cases} A & \text{if } B \in F_1 \\ \emptyset & \text{if } B \in F_0 \end{cases}$$

Thus $P_0(A \times B) = P_1(A \times B)^c \in E$ and the theorem is proved.

Remark 1.4. Theorem 1.3 extends to cases when partitions of E or E and F are taken simultaneously. Our operation extends a construction by Muller *et al.* (1992).

2. ON THE PROBLEM OF BEST EXTENSION OF SIGNED MEASURES ON FINITE SET LOGICS

Let (E, X) be a finite c.l. such that every signed measure (s.m.) v on E can be extended as a s.m. $v^{\sim}$ on the algebra $P(X)$ of all subsets of X. Thus, there exists a (not unique) function f: $X \to R$ such that

$$\tilde{v}(A) = v_f(A) = \sum_{x \in A} f(x) \qquad (***)$$

Hence, for two extensions v_f, v_g the problem of comparison arises. For the simple calculation by formula (***) we must consider the extension v_f "better" than v_g if the function f has more equal values than g. Now we give the exact definition.

Let $n \in \mathbf{N}$, $R(n) = \{(n_1, n_2, \ldots, n_k) | k, n_i \in \mathbf{N}, n_1 \geq n_2 \geq \cdots \geq n_k, n_1 + n_2 + \cdots + n_k = n\}$. Consider the relation $\succ$ on the set $R(n)$:

$$(n_1, n_2, \ldots, n_k) \succ (m_1, m_2, \ldots, m_k) \text{ iff}$$

(a) If $k < j$, then $n_1 \geq m_1$.

(b) If $k = j$, then $\succ$ coincide with the lexicographic relation.

Then this relation is a partial order relation and $(R(n), \succ)$ is a lattice.

Let now $E(v)$ be the set of all extensions of the s.m. v to the c.l. (E, X); v_f, $v_g \in E(v)$. We suppose that $f = \sum_{i=1}^{k} \lambda_i I_{A_i}$, $\lambda_i \neq \lambda_s$, $A_i \cap A_s = \emptyset$ $(i \neq s)$, $\bigcup_{i=1}^{k} A_i = X$, and card $A_1 \geq$ card $A_2 \geq \cdots \geq$ card A_k. Also we suppose the same for the function $g = \sum_{s=1}^{j} \mu_s I_{B_s}$. The (card A_1, card A_2, ..., card A_k), (card B_1, card B_2, ..., card B_j) $\in R$(card X).

Definition 2.1. For two extensions v_f, v_g of the signed measure v on the finite concrete logic (E, X) we put

$$v_f \geq v_g \quad \text{iff} \quad (\text{card } A_1, \text{card } A_2, \ldots, \text{card } A_k) \succ (\text{card } B_1, \text{card } B_2, \ldots, \text{card } B_j)$$

in the lattice R(card X).

It is easy to see that $(E(v), \geq)$ is a partially ordered set. In the example below we find the best extensions for some states.

Example 2.2. Consider the set logic E_6, whose Greechie diagram is a 6-polygon with three atoms on each edge. Then E_6 is absolutely regular (Sultanbekov, 1993). We examine one of the ten minimal representations for

E_6 as a concrete logic. Let us denote by $P_0, P_1, \ldots, P_5$ the vertices and $Q_0, Q_1, \ldots, Q_5$ the middle atoms of E_6. Put $X = \{a_0, a_1, \ldots, a_5, d_0, d_1, d_2, e\}$ and $\tilde{P}_k = \{a_{k-1}, a_{k+1}, d_k\}$, $\tilde{Q}_k = X \backslash \tilde{P}_k \cup \tilde{P}_{k+1}$ (always indices of a_k are modulo 6 and indices of d_k are modulo 3), $\tilde{E}_6 = \{\tilde{P}_k, \tilde{Q}_k | k = 0, 1, \ldots, 5\}$.

Applying Theorem 1.5 of Sultanbekov (1993), we immediately have that dim $\tilde{E}_6(v) = 3$ for all s.m. v on the $(\tilde{E}_6, X)$. For example, denote by c_0, b_0 the two-valued states on E_6 determined by $c_0(P_k) = 1$ (k is even), $c_0(P_k) = 0$ (k is odd), $b_0(P_0) = 1$, $b_0(P_k) = 0$ ($k = 1, \ldots, 5$). Then for the extension f of the c_0 we have $f(e) = -0.5$,

$$f(a_k) = 0.5[f(d_k) - f(d_{k+1}) - f(d_{k+2})] \qquad (k \text{ is even})$$

$$f(a_k) = 0.5[1 + f(d_k) - f(d_{k+1}) - f(d_{k+2})] \qquad (k \text{ is odd})$$

Calculations show that the best extension of c_0 is the unique function $f = 0.5[I_{\{a1,a3,a5\}} - I_{\{e\}}]$. Analogously for the b_0 we find the best extension which is the unique function $g = 0.5[I_{\{a1,a5,e\}} - I_{\{a3\}}]$.

3. OPEN PROBLEMS

Problem 3.1. Let E, F, G be c.l. and F be isomorphic to G ($F \approx G$). Consider a partition $p = \{F_0, F_1\}$ of the logic F. Does there exist a partition $q = \{G_0, G_1\}$ of the logic G such that $E \bowtie F_p \approx E \bowtie G_q$?

Problem 3.2. Let M be a c.l. on the $X \times Y$ which contains all $A \times B$, $A \in E$, $B \in F$, where E, F are c.l.s on the set X and set Y, respectively. What are necessary and sufficient conditions for $M \approx E \bowtie F_p$?

Problem 3.3. Let (E, X), (F, Y) be c.l.s. When does $\cap \{E_p \bowtie F_q | p, q$ are partitions in E and $F\}$ coincide with the least c.l. containing all sets of the form $A \times B$ ($A \in E$, $B \in F$)?

Problem 3.4. Let (E, X), (F, Y) be c.l.s and p a partition of the F. Suppose that the v, w are s.m.s on E and F, respectively. Define a s.m. $v \bowtie w$ on the c.l. $E \bowtie F_p$.

Problem 3.5. Suppose that Problem 3.4 has been solved and v_f, w_g are maximal elements of $E(v)$ and $E(w)$. Is $v_f \bowtie w_g$ a maximal element of $E(v \bowtie w)$?

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